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ON PARTIALLY CONFORMAL QC DEFORMATIONS

HIROMI OHTAKE

Kyoto University of Education
大竹 博巳 (京都教育大)

1. Let $M(R)$ be the Banach space of all Beltrami differentials $\mu = \mu(z)\dfrac{d\bar{z}}{dz}$ on a Riemann surface R with norm $\|\mu\|_\infty := \operatorname{ess\,sup}|\mu(z)|$. We denote by $M(R)_1$ the open unit ball of $M(R)$. Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. For each $\mu \in M(\mathbb{D})_1$, there is a unique normalized quasiconformal self-mapping W^μ of $\mathbb{D}$ whose Beltrami coefficent $\mu(W^\mu) := W^\mu_{\bar{z}}/W^\mu_z$ is μ, that is, $W^\mu\colon \mathbb{D} \to \mathbb{D}$ is a homeomorphism whose generalized derivatives satisfy the Beltrami equation $f_{\bar{z}} = \mu f_z$, and its continuous extension to the closed unit disk $\overline{\mathbb{D}}$ fixes 1, i and -1. Two elements μ and ν in $M(\mathbb{D})_1$ are said to be *equivalent* if W^μ and W^ν have the same boundary values. Let R be a hyperbolic Riemann surface and $\pi\colon \mathbb{D} \to R$ be a universal covering mapping. We define μ, $\nu \in M(R)_1$ are equivalent when so are their pull-backs $\pi^*\mu$ and $\pi^*\nu$, and quasiconformal mappings $f\colon R \to f(R)$ and $g\colon R \to g(R)$ are equivalent if so are their Beltrami coefficients $\mu(f)$ and $\mu(g)$. It is known that f and g are equivalent if and only if there is a conformal mapping $h\colon f(R) \to g(R)$ such that $h \circ f$ is homotopic to g modulo the border of R. The Teichmüller space $T(R)$ of R is the quotient space of $M(R)_1$ with respect to this equivalence relation. We denote by $[\mu]$ the equivalence class containing μ, and identify it with the marked Riemann surface $[f(R), f]$, $\mu(f) = \mu$.

Let V be a measurable subset of R and set

$$M(V)_1 := \{\mu \in M(R)_1\colon \mu|_{R\setminus V} = 0\}.$$

A quasiconformal mapping f is 'conformal' outside V if $\mu(f) \in M(V)_1$, so we say $[f(R), f]$ is a partially conformal qc deformation of $[R, \mathrm{id}_R]$. A family of partially conformal qc mapings is useful to investigate Teichmüller spaces and extremal problems on them (see for example Krushkal [5], Gardiner [2], [3], Reich [10] and Fehlmann-Sakan [1]).

2. We summarize some known facts. First of all, in general, $[M(V)_1] \neq T(R)$ (cf. Savin [11]). For example, if $R \setminus V$ is an incompressible annular domain, then $[M(V)_1] \neq T(R)$. But if $R \setminus V$ is a topological disk, then $[M(V)_1] = T(R)$.

If R is of finite conformal type, that is, R is a Riemann surface obtained by removing a finite number of punctures from a compact one, then $[M(V)_k]$ is a neiborhood of the origin [0] of $T(R)$ for any V with positive measure and any $0 < k \leq 1$. This is a classical result. While there are R of infinite conformal type and a subset V of R with positive measure such that $[M(V)_1]$ is not a neiborhood of [0] (Oikawa [9]).

A general necessary condition for V to insure that $[M(V)_1]$ becomes a neiborhood of [0] is

$$r(V) := \inf\left\{ \iint_V |\phi|\, dxdy \colon \phi \in A_2^1(R), \|\phi\|_1 = 1 \right\} > 0. \tag{1}$$

Moreover, when $R = \mathbb{D}$, the condision (1) is equivalent to a simple geometric one:

$$\inf\{\mathrm{Area}(V \cap \Delta(z;\rho)) \colon z \in \mathbb{D}\} > 0 \qquad \text{for some } \rho > 0$$

where $\Delta(z;\rho)$ is the hyperbolic disk with center z and radius ρ, and Area means its hyperbolic area (Ohtake [7]).

On the other hand, a known sufficient condition is as follows. Set

$$\omega(z) := \sup\{\lambda(z)^{-2}|\phi(z)| \colon \phi \in A_2^1(R), \|\phi\|_1 = 1\}.$$

It is not difficult to see that the function ω on R is continuous and vanishing at the punctures of R. If V has positive measure and if

$$\iint_V \max\{\omega(z)^2, 1\}\, dxdy < \infty,$$

then $[M(V)_k]$ contains [0] in its interior for any $0 < k \leq 1$ (Ohtake [6]).

3. We give here a quantative version of the necessary condition (1) above.

Theorem 1. *If $[M(R)_k] \subset [M(V)_{k'}]$, then we have*

$$r(V) \geq \frac{K-1}{K'-1}. \tag{2}$$

where $K := (1-k)/(1+k)$, $K' := (1-k')/(1+k')$.

Proof. Take arbitrary $0 < t < k$ and $\phi \in A_2^1(R)$ with $\|\phi\|_1 = 1$. Let $f_0 \colon R \to R_0$ be a quasiconformal mapping whose Beltrami coefficient is $t\overline{\phi}/|\phi|$ and $\psi \in A_2^1(R_0)$ be the terminal differential of the Teichmüller mapping f_0 (cf. Lehto [4]). Then $f_0^{-1} \colon R_0 \to R$ is a Teichmüller mapping with $\mu(f_0^{-1}) = -k\overline{\psi}/|\psi|$. By asummption, there is a quasiconformal mapping $f \colon R \to R_0$ which is equivalen to f and whose Beltrami coefficient $\mu(f)$ is in $M(V)_{k'}$. Applying Reich-Strebel inequality (Strebel [12], [13]) to $-\psi$ and $f \circ f_0^{-1}$ equivalent to the identity mapping of R_0, we have

$$\|\psi\|_1 \leq \iint_{R_0} |\psi| \frac{\left|1 + \mu(f \circ f_0^{-1})\psi/|\psi|\right|^2}{1 - |\mu(f \circ f_0^{-1})|^2}\, dudv.$$

Since

$$K(f_0)|\phi(z)|\,dxdy = |\psi(w)|\,dudv, \qquad w = f_0(z)$$
$$\frac{\overline{\psi}(w)}{|\psi(w)|} = \frac{p(z)}{\overline{p}(z)} \cdot \frac{\overline{\phi}(z)}{|\phi(z)|}, \qquad p = (f_0)_{\overline{z}}$$
$$\mu(f \circ f_0^{-1})(w) = \frac{\mu(f)(z) - \mu(f_0)(z)}{1 - \overline{\mu}(f_0)(z)\mu(f)(z)} \cdot \frac{p(z)}{\overline{p}(z)},$$

change of variable gives us

$$\begin{aligned}
K(f_0) &\leq K(f_0) \iint_R \frac{\left|1 - \mu(f_0)\dfrac{\phi}{|\phi|}\right|^2 \left|1 + \mu(f)\dfrac{\phi}{|\phi|} \cdot \dfrac{1 - \overline{\mu}(f_0)\overline{\phi}/|\phi|}{1 - \mu(f_0)\phi/|\phi|}\right|^2}{(1 - |\mu(f_0)|^2)(1 - |\mu(f)|^2)} |\phi|\,dxdy \\
&= \iint_R \frac{\left|1 + \mu(f)\phi/|\phi|\right|^2}{1 - |\mu(f)|^2} |\phi|\,dxdy \\
&\leq K' \iint_V |\phi|\,dxdy + \iint_{R \setminus V} |\phi|\,dxdy \\
&= (K' - 1) \iint_V |\phi|\,dxdy + 1.
\end{aligned}$$

Letting $t \to k$, we have a desired inequality (2). $\square$

We can show a partial converse of Theorem 1. Its proof and the details are omitted and will appear elsewhere.

Theorem 2. *For $A > 0$ and $l > 0$, there are positive constants C and $t_0 \leq 1$ such that if a Riemann surface R has hyperbolic area less than A and if the length of each closed geodesic of R is not shorter than l, then*

$$[M(R)_t] \subset [M(V)_{Ct/r(V)^2}] \qquad \text{for any } 0 \leq t \leq t_0.$$

where the constants C and t_0 depend only on A, l and $r(V)$ but not on R nor V.

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1 FUKAKUSA-FUJINOMORI, FUSHIMI, KYOTO 612 JAPAN
E-mail address: h54378@sakura.kudpc.kyoto-u.ac.jp